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"A study of Vertex - Edge Coloring Techniques with Application"

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Abstract

Graph coloring is one of the most important concepts in graph theory and is used in many real time applications in engineering & science; various coloring methods are available and can be used on requirement basis. The proper coloring of a graph is the coloring of the vertices and edges with minimal number of colors such that no two vertices should have the same color. The minimum number of colors is called as the chromatic number and the graph is called properly colored graph. This paper present the importance of graph coloring in various fields.

Keywords: Bipartitite graph

History of Graph theory:

The origin of graph theory started with the problem of Koinsber bridge, in 1735. This problem lead to the concept of Eulerian Graph. Euler studied the problem of Koinsberg bridge and constructed a structure to solve the problem called Eulerian graph. In 1840, A.F Mobius gave the idea of complete graph and bipartite graph and Kuratowski proved that they are planar by means of recreational problems. The concept of tree, (a connected graph without cycles[7]) was implemented by Gustav Kirchhoff in 1845, and he employed graph theoretical ideas in the calculation of currents in electrical networks or circuits. In 1852, Thomas Gutherie found the famous four color problem. Then in 1856, Thomas. P. Kirkman and William R.Hamilton studied cycles on polyhydra and invented the concept called Hamiltonian graph by studying trips that visited certain sites exactly once. In 1913, H.Dudeney mentioned a puzzle problem. Eventhough the four color problem was invented it was solved only after a century by Kenneth Appel and Wolfgang Haken. This time is considered as the birth of Graph Theory.Caley studied particular analytical forms from differential calculus to study the trees. This had many implications in theoretical chemistry. This lead to the invention of enumerative graph theory. Any how the term "Graph" was introduced by Sylvester in 1878 where he drew an analogy between "Quantic invariants" and co- variants of algebra and molecular diagrams. In 1941, Ramsey worked on colorations which lead to the identification of another branch of graph theory called extremel graph theory. In 1969, the four color problem was solved using computers by Heinrich. The study of asymptotic graph connectivity gave rise to random graph theory.

Introduction

Graph Coloring:

Graph coloring is one of the most important concepts in graph theory and is used in many real time applications

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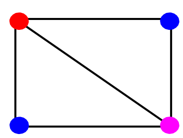
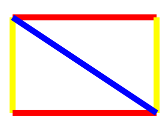
in various fields. Various coloring methods are available and can be used on requirement basis. The proper coloring of a graph is the coloring of the vertices and edges with minimal number of colors such that no two vertices should have the same color. The minimum number of colors is called as the chromatic number and the graph is called properly colored graph

Proper vertex coloring with Chromatic number 3 Proper edge coloring with Chromatic number 3

VERTEX COLOURINGS

k- Vertex colouring:

A k-vertex colouring of a graph G is an assignment of k colours,1,2,…,k, to the vertices of G. The colouring is proper if no two distinct adjacent vertices have the same colour.

k- Vertex colorable:

A graph G is k-vertex colorable if G has a proper k-vertex colouring. k- vertex colorable is also called as k-colorable. A graph is k-colorable if and only if its underlying simple graph is k-colorable.

Uniquely k-colorable:

A graph G is called uniquely k-colorable if any two proper k- colourings of G induce the same partition of V.

Chromatic number $\chi(G)$:

The chromatic number $\chi(G)$,of a graph G is the minimum k for which, G is k-colorable.

K- Chromatic:

If $\chi(G)=k$, G is said to be k-chromatic.

Critical graph:

A graph G is critical if $\chi(H)<\chi(G)$ for every proper sub graph H of G. Such graphs were first investigated by Dirac (1952).

k-critical:

A k-critical graph is one that is k-chromatic and critical.Every k- chromatic graph has a k-critical subgraph.

Theorem 1.30:

If G is k-critical,then $\delta(G)\geq k-1$.

Theorem1.32:

In a critical graph,no vertex cut is a clique.

Theorem 1.33 (Brook 1973):

If G is a connected simple graph and is neither an odd cycle nor a complete graph, then $\chi(G)\leq\Delta(G)$.

Chromatic polynomial f(G,t):

The chromatic polynomial f(G,t) of a graph G is the number of different colourings of a labeled graph G that use t (or) fewer colours(like permutation and combination).

Theorem 1.34:

If G is a tree with n vertices then $f(G,t)=t(t-1)^{n-1}$.

APPLICATIONS OF VERTEX COLOURINGS

A Storage Problem:

A company manufactures n chemicals $C_1,C_2,...,C_n$.Certain pairs of these chemicals are incompatible and would cause explosions if brought into contact with each other. As a precautionary measure the company wishes to partition its warehouse into

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compartments, and store incompatible chemicals in different compartments. What is the least number of compartments into which the warehouse should be partitioned? This is known as the Storage Problem. We obtain a graph on the vertex set $\{v_1,v_2,...,v_n\}$ by joining two vertices v_i and v_j if and only if the chemicals C_i and C_j are incompatible. It is easy to see that the least number of compartments into which the warehouse should be partitioned is equal to the chromatic number of G.

EDGE COLOURINGS

K-edge colouring:

A k-edge colouring of a loop less graph G is an assignment of k colours,1,2,…,k, to the edges of G. The colouring is proper if no two distinct adjacent edges have the same colour.

K-edge colorable:

A graph G is k-edge colorable if G has a proper k-edge colouring. Edge chromatic number $\chi'(G)$:

The edge chromatic number $\chi'(G)$,of a loop less graph G is the minimum k for which G is k-edge colorable.

K-edge chromatic:

If $\chi'(G)=k$,G is said to be k-edge chromatic.

Let ℓ be a given k-edge colouring of G. We shall denote by c(v) the number of distinct colors represented at v. Clearly, we always have $c(v)\leq d(v)\rightarrow(2)$. Moreover, ℓ is a proper k-edge colouring if and only if equality holds in (2) for all vertices in G.

We shall call a k-edge colouring ℓ' an improvement on ℓ if $\sum_{v\in V}c'(v)>\sum_{v\in V}c(v)$ where c'(v) is the number of distinct colors represented at v in the colouring ℓ'.

Optimal k-edge colouring:
An optimal k-edge colouring is one which cannot be improved.
Theorem 1.35:
If G is bipartite, then $\chi'(G)=\Delta(G)$.
Theorem 1.36 (Vizing 1964):
If G is simple, then $\chi'(G)=\Delta(G)$ or $\chi'(G)=\Delta(G)+1$.
Uniquely k-edge colorable:
A graph G is called uniquely k-edge colorable if any two proper k- edge colorings of G induce the same partition of E.

APPLICATIONS OF EDGE COLOURINGS

The Timetabling Problem:

In a school there are m teachers $X_1,X_2,...,X_m$ and n dasses $Y_1,Y_2,...,Y_n$.Given the teacher X_i is required to teach class Y_j for p_{ij} periods, schedule a complete timetable in the minimum possible number of periods. This is known as the Timetabling Problem. We represent the teaching requirements by a bipartite graph G with bipartition (X,Y),where $X=\{x_1,x_2,...,x_n\}$ and $Y=\{y_1,y_2,..., y_n\}$ and vertices x_i and y_j are joined by p_{ij} edges. Now in any one period, each teacher can teach at most one class, and each class can be taught by at most one teacher. This is at least our assumption. Thus a teaching schedule for one period corresponds to a matching in the graph and, conversely, each matching corresponds to a possible assignment of teachers to classes for one period. Our problem, therefore is to partition the edges of G into as few matching as possible or, equivalently, to properly colour the edges of G with as few colors as possible. Since G is bipartite, we know by theorem 1.35 $\chi'(G)=\Delta(G)$.Hence, if no teacher teaches for

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more than p periods, the teaching requirements can be scheduled in a p-period timetable. We thus have a complete solution to the problem.

Graph coloring techniques in scheduling:

Here some scheduling problems that uses variants of graph coloring methodologies such as precoloring, list Coloring, multicoloring, minimum sum coloring are given in brief.

Job scheduling:

Here the jobs are assumed as the vertices of the graph and there is an edge between two jobs if they cannot be executed simultaneously. There is a 1-1 correspondence between the feasible scheduling of the jobs and the colorings of the graph. [4]

Aircraft scheduling:

Assuming that there are k aircrafts and they have to be assigned n flights. The i^{th} flight should be during the time interval (a_i, b_i). If two flights overlap, then the same aircraft cannot be assigned to both the flights. This problem is modeled as a graph as follows.

The vertices of the graph correspond to the flights. Two vertices will be connected, if the corresponding time intervals overlap. Therefore, the graph is an interval graph that can be colored optimally in polynomial time. [4]

Bi-processor tasks:

Assume that there is a set of processors and set of tasks. Each task has to be executed on two processors simultaneously and these two processors must be pre assigned to the task. A processor cannot work on two jobs simultaneously. This type of tasks will arise when scheduling of file transfers between processors or in case of mutual diagnostic besting of processors. This can be modeled by considering a graph whose vertices correspond to the processes and if there is any task that has to be executed on processors i and j, then and edge to be added between the two vertices. Now the scheduling problem is to assign colors to edges in such a way that every color appears at most o nce at a vertex.

If there are no multiple edges in the graph (i.e) no two tasks require the same two processors then the edge coloring technique can be adopted. The authors have developed an algorithm for multiple edges which gives an 1-1 approximate solution. [4]

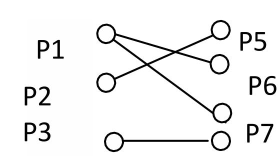

Figure – 3 Tasks allocated to processors

The diagram shows the tasks namely task1, task2, task3 and task4 are allocated to the processors (P1, P5); (P1, P6); (P2, P4) and (P3, P7) respectively

Precoloring extension:

In certain scheduling problems, the assignments of jobs are already decided. In such cases precoloring technique can be adopted. Here some vertices of the graph will have preassigned color and the precoloring problem

has to be solved by extending the coloring of the vertices for the whole graph using minimum number of colors. [4].

List coloring:

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In list coloring problem, each vertex v has a list of available colors and we have to find acoloring where the color of each vertex is taken from the list of available colors. This list coloring can be used to model situations where a job can be processed only in certain time slots or can be processed only by certain machines.[4]

Minimum sum coloring:

In minimum sum coloring, the sum of the colors assigned to the vertices is minimal in the graph. The minimum sum coloring technique can be applied to the scheduling theory of minimizing the sum of completiontimes of the jobs. The multicolor version of the problem can be used to model jobs with arbitrary lengths. Here, the finish time of a vertex is the largest color assigned to it and the sum of coloring is the sum of the finish time of the vertices. That is the sum of the finish times in a multicoloring is equal to the sum of completion times in the corresponding schedule.[4]

Time table scheduling:

Allocation of classes and subjects to the professors is one of the major issues if the constraints are complex.Graph theory plays an important role in this problem. For m professors with n subjects the available number of p periods timetable has to be prepared. This is done as follows.

Figure –4

p	n_1	n_2	n_3	n_4	n_5
m_1	2	0	1	1	0
m_2	0	1	0	1	0
m_3	0	1	1	1	0
m_4	0	0	0	1	1

A bipartite graph (or bigraph is a graph whose vertices can be divided into two disjoint sets *U* and *V* such that every edge connects a vertex in *U* to one in *V*; that is, *U* and *V* are independent sets[7]) G where the vertices are the number of professors say m_1, m_2, m_3, m_4, ……. m_k and n number of subjects say n_1, n_2, n_3, n_4, ……. n_m such that the vertices are connected by p_i edges. It is presumed that at any one period each professor can teach at most one subject and that each subject can be taught by maximum one professor. Consider the first period. The timetable for this single period corresponds to a matching in the graph and conversely, each matching corresponds to a possible assignment of professors to subjects taught during that period. So, the solution for the timetabling problem will be obtained by partitioning the edges of graph G into minimum number of matching. Also the edges have to be colored with minimum number of colors. This problem can also be solved by vertex coloring algorithm. " The line graph L(G) of G has equal number of vertices and edges of G and two vertices in L(G) are connected by an edge iff the corresponding edges of G have a vertex in common. The line graph L(G) is a simple graph and a proper vertex coloring of L(G) gives a proper edge coloring of G by the same number of colors. So, the problem can be solved by finding minimum proper vertex coloring of L(G)." For example, Consider there are 4 professors namely m_1, m_2, m_3, m_4,. and 5 subjects say n_1, n_2, n_3, n_4, n_5 to be taught. The teaching requirement matrix p = [p_{ij}] is given below.

The teaching requirement matrix for four professors and five subjects.

Figure 5

Bipartite graph with 4 professors and 5 subjects

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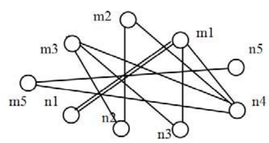

Finally, the proper coloring of the above mentioned graph can be done by 4 colors using the vertex coloring algorithm which leads to the edge coloring of the bipartite

multigraph G.

Map coloring and GSM mobile phone networks:

Groups Special Mobile (GSM) is a mobile phone network where the geographical area of this network is divided into hexagonal regions or cells. Each cell has a communication tower which connects with mobile phones within the cell. All mobile phones connect to the GSM network by searching for cells in the neighbors. Since GSM operate only in four different frequency ranges, it is clear by the concept of graph theory that only four colors can be used to color the cellular regions. These four different colors are used for proper coloring of the regions. Therefore, the vertex coloring algorithm may be used to assign at most four different frequencies for any GSM mobile phone network.

Conclusion:

The main aim of this paper is to present the importance of graph theoretical ideas in various areas of compute applications for researches that they can use graph theoretical concepts for the research. An overview is presented especially to project the idea of graph theory. So, the graph theory section of each paper is given importance than to the other sections.

Researches may get some information related to graph theory and its applications in computer field and can get some ideas related to their field of research.

References:

[1] Anindya J.Pal, Samar S.Sarma, Biman Ray, "CCTP, Graph Coloring algorithms – Soft computing Solutions IEEE, 2007

[2] Daniel Marx, "Graph Coloring problems and their applications in scheduling",

[3]Narasingh Deo, "Graph theory with applications to engineering and computer science", Prentice Hall of India, 1990

[4] Shariefuddin Pirzada and Ashay Dharwadker, "Journal of the Korean Society for Industrial and applied Mathematics, Volume 11, No.4,2007

[5] http://www.dharwadker.org/vertex_cover/

[6]http://www.icaen.uiowa.edu/~dip/LECTURE/Understanding6.html

[7 wj:math.sfsu.edu/beck/teach

[8] wikipedia.org/wiki/Median graph

[9] http://en.wikipedia.org/wiki/Bipartite_grap